Splitting the K-Terminal Reliability

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Abstract

Let $G = (V, E)$ be a graph and $K \subseteq V$ a set of terminal vertices. Assume now that the edges of G are failing independently with given probabilities. The K-terminal reliability $R(G, K)$ is the probability that all vertices in K are mutually connected.

In this article we propose an efficient splitting formula for $R(G, K)$ at a separating vertex set of G by lattice theoretic methods.

Keywords: K-terminal reliability, Möbius inversion, partition lattice, join matrices, splitting

1 Introduction

Let $G = (V, E)$ be a graph and $K \subseteq V$ a set of terminal vertices. Assume now that the edges of G are failing independently with given probabilities. The K-terminal reliability $R(G, K)$ is the probability that all terminal vertices are mutually connected in G.

Ball [3] shows that the computational complexity of $R(G, K)$ is NP-hard for arbitrary graphs. In the case of series parallel graphs Wood [19] proposed a polynomial time algorithm for the computation of $R(G, K)$ by polygon-to-chain reductions.

A decomposition (G^1, G^2, X) of G consists of two subgraphs G^1 and G^2, so that $G^1 \cup G^2 = G$ and $G^1 \cap G^2 = (X, \emptyset)$. Note that X is a separating vertex set of G. In this article we propose a scheme for the computation of $R(G, K)$ given a decomposition, pursuing the ideas of Rosenthal [13].

Bienstock [5] and Tittmann [17] examine such decomposition methods by utilising the lattice of set partitions of X. Nice results are especially derived when $K = V$ is assumed, but are unsatisfactory in the general case.

The centrepiece of this article is Theorem 55 representing $R(G, K)$ by the linear combination

$$R(G, K) = \sum_{\pi, \sigma \in \Pi_l(X, \pi_X)_0} R(G^1_\sigma, K^1_\sigma) f(\sigma, \pi) R(G^2_\pi, K^2_\pi), \tag{1}$$

where G^1_π and G^2_σ are emerging from the subgraphs G^1 and G^2 by a identification of vertices. Our result is therefore a generalisation of the result given by Bienstock [6].

We emphasise that there are two main advantages of our approach compared to previously proposed methods for the general K-terminal reliability by Rosenthal [13] and Bienstock [5].

The first advantage is the small cardinality of the state set $\Pi_l(X, \pi_X)_0$ in Equation 1. We show that if X is a vertex separator of cardinality n, the state set can have at most $B(n+1)-1$ different elements. Here $B(n)$ is the n-th Bell number denoting the number of set partitions of an n-element set. We mention that the number of possible states might be even more reduced, if the separating vertex set contains terminal vertices. Hence we are able to compute $R(G, K)$ even in the case of separating vertex sets, that were not accessible by former methods.

The second advantage of the new decomposition formula is the neat symmetry in its representation, which allows a recursive application by the transfer-matrix method, which is not presented here for the sake of brevity.

2 Partially Ordered Sets

This section compiles some necessary definitions concerning partially ordered sets or short posets.

Definition 1. Denote by $(P, \leq_P)$ a poset of a finite set P. If P has a maximum or minimum then it is denoted by 1 or 0, respectively. Given two elements $x, y \in P$, then

$[x, y]_P := \{z \in P : x \leq z \leq y\}$ is an *interval* in P. Given any subset $Q \subseteq P$, we say that $(Q, \leq_Q)$ is a *subposet* of P if for all $p, q \in Q$ we have $p \leq_Q q$ if and only if $p \leq_P q$. Observe that every interval I of a poset P is a subposet of P.

Definition 2. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets. The *product order* $(P \times Q, \leq_{P\times Q})$ consists of all ordered pairs in $P \times Q$, where $(p, q) \leq_{P\times Q} (r, s)$ if and only if $p \leq_P r$ and $q \leq_Q s$.

Definition 3. Let $(P, \leq_P)$ be a poset and $p, q \in P$. We say that $u \in P$ is an *upper bound* of p and q if $p \leq u$ and $q \leq u$ and if every other upper bound $s \in P$ of p and q satisfies $u \leq s$, we say that u is the *smallest upper bound* $u = p \vee q$ of p and q. The notion of *lower bound* and the *greatest lower bound* $p \wedge q$ of p and q is defined likewise.

Definition 4. A *lattice* $(L, \vee, \wedge)$ is a poset $(L, \leq)$, so that for all $p, q \in L$ the elements $p \vee q$ and $p \wedge q$ exist. In the case that we only demand that $p \vee q$ exists for all $p, q \in L$ we say that $(L, \vee)$ is an *upper semilattice*.

Definition 5. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets. A function $f\colon P \to Q$ is *order preserving* if for all $p, q \in P$ we have $f(p) \leq_Q f(q)$ if and only if $p \leq_P q$. The posets P and Q are *isomorphic*, if there is an order preserving and bijective function $f\colon P \to Q$, and we write $P \simeq Q$.

3 The Incidence Algebra

This section states some of the definitions and results concerning incidence algebras of posets. Rota [14] applies the incidence algebra of posets in combinatorics and Crapo [7] contributes the versatile Theorem 11. Finally, we mention that Aigner [1] gives a compilation of results, that are utilising incidence algebras in enumerative combinatorics.

Definition 6. Let $(P, \leq)$ be a poset. We denote by $I(P)$ the set of all functions $f\colon P \times P \to \mathbb{R}$ with $f(x, y) = 0$, whenever $x \not\leq y$ holds. For every $f, g \in I(P)$ define the *convolution product* $f \star g \in I(P)$ by

$$(f \star g)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y). \tag{2}$$

The set $I(P)$ endowed with the pointwise addition, multiplication with scalars $\lambda \in \mathbb{R}$, and the convolution product is the *incidence algebra $I(P)$ of P*.

Definition 7. Let P be a poset and $x, y \in P$. The incidence functions

$$\zeta_P(x, y) = \begin{cases} 1 & x \leq y \\ 0 & \text{else} \end{cases} \quad \text{and} \quad \delta_P(x, y) = \begin{cases} 1 & x = y \\ 0 & \text{else} \end{cases} \tag{3}$$

are the *Zeta-function* and the *Delta-function* of P.

Definition 8. Let P be a poset. The unique incidence function $\mu_P \in I(P)$, that satisfies the equation $\mu_P \star \zeta_P = \delta_P$, is the *Möbius function of* P.

Proposition 9 (Rota [14])**.** *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets. The Möbius function of the product order $(P \times Q, \leq_{P\times Q})$ satisfies*

$$\mu_{P\times Q}((p,q),(r,s)) = \mu_P(p,r)\mu_Q(q,s) \tag{4}$$

for all $(p,q),(r,s) \in P \times Q$.

Definition 10. Let L be a lattice with minimum 0 and maximum 1. L is *complemented* if for all $p \in L$ there is a $q \in L$ with $p \vee q = 1$ and $p \wedge q = 0$.

Theorem 11 (Crapo [7])**.** *Let L be a finite lattice, that is not complemented. Then $\mu_L(0,1) = 0$.*

4 Labelled Set Partitions

In this section we introduce the lattice of labelled set partitions $\Pi_l(X)$ of a finite set X and determine its Möbius function. The study of this lattice is helpful when considering the splitting of the K-terminal reliability. First approaches in this direction are made by Bienstock [5] and Tittmann [16].

Definition 12. Let X be a finite set. A *set partition* $\pi = \{B_1, \ldots, B_k\}$ of X is a collection of mutually disjoint and non-empty subsets of X, the *blocks*, with union X. The set of all set partitions of X is denoted by $\Pi(X)$.

We define the poset $(\Pi(X), \leq)$ by setting $\sigma \leq \pi$ if every block of σ is a subset of a block in π for all $\sigma \in \pi \in \Pi(X)$. Note that $(\Pi(X), \leq)$ is a lattice with minimum $\hat{0}$ and maximum $\hat{1}$.

Finally, we mention that the number of all set partitions of an n-element set are the *Bell numbers* $B(n)$ and the number of all set partitions of an n-element set with k blocks the *Stirling numbers of the second kind* $S(n,k)$.

Theorem 13 (Rota [14])**.** *Let X be a non-empty n-element set. Then the Möbius function in $\Pi(X)$ satisfies*

$$\mu_{\Pi(X)}(\hat{0},\hat{1}) = (-1)^{n-1}(n-1)!. \tag{5}$$

Definition 14. Let X be a finite set and $l \notin X$ a distinguished label element. A *labelled set partition* π is a collection of subsets $\pi = \{B_1 \cup L_1, \ldots, B_k \cup L_k\}$ with $B_i \subseteq X$ and $L_i \in \{\emptyset, \{l\}\}$, so that $\{B_1, \ldots, B_k\}$ is a set partition of X. The set of all labelled set partitions is denoted by $\Pi_l(X)$. A block $B_i \cup L_i \in \pi$ is *unlabelled* if $L_i = \emptyset$ and *labelled* if $L_i = \{l\}$.

For convenience of display we use the notation $\pi = B_1L_1|\ldots|B_kL_k$ and we drop all unnecessary parentheses. For example we write $\pi = 12l|3|45$ instead of $\pi = \{\{1,2,l\},\{3\},\{4,5\}\}$.

Definition 15. Let $\pi \in \Pi_l(X)$ with $\pi = B_1L_1|\dots|B_kL_k$ and $Y \subseteq X$. We say that $\pi \sqcap Y \in \Pi_l(X)$ is the *restriction of* π *to* Y with

$$\pi \sqcap Y = \bigcup_{\substack{B_iL_i \in \pi \\ B_i \cap Y \neq \emptyset}} \{(B_i \cap Y) \cup L_i)\}. \tag{6}$$

Definition 16. Let $\sigma, \pi \in \Pi_l(X)$ and set $\sigma \leq \pi$ if every block of σ is a subset of a block in π. Observe that $(\Pi_l(X), \leq)$ is a poset with minimum $\hat{0}_l$ and maximum $\hat{1}_l$. It can be shown that $(\Pi_l(X), \leq)$ is even more a lattice $(\Pi_l(X), \vee, \wedge)$.

Proposition 17. *Let $\sigma, \pi \in \Pi_l(X)$ with $\sigma \leq \pi$ and $\pi = B_1L_1|\dots|B_kL_k$. Then the interval $[\sigma, \pi]_{\Pi_l(X)}$ is isomorphic to the k-fold product order*

$$\prod_{i=1}^{k} [\sigma \sqcap B_i, B_iL_i]_{\Pi_l(B_i)}. \tag{7}$$

Lemma 18. *Let $\pi \in \Pi_l(X)$ be a labelled set partition with at least one labelled and at least one unlabelled block. Then $\mu_{\Pi_l(X)}(\pi, \hat{1}_l) = 0$.*

Proof. We can assume without loss of generality that $\pi = B_1L_1|B_2L_2|\dots|B_kL_k$ with $k \geq 2$, so that $L_1 = \emptyset$ and $L_2 = \{l\}$. Consider now the labelled set partition $\tilde{\pi} \in \Pi_l(\{1,\dots,k\})$ with $\tilde{\pi} = 1L_1|2L_2|\dots|kL_k$. Then we have

$$[\pi, \hat{1}_l]_{\Pi_l(X)} \simeq [\tilde{\pi}, \hat{1}_l']_{\Pi_l(\{1,\dots,k\})},$$

where $\hat{1}_l'$ denotes the maximum in $\Pi_l(\{1,\dots,k\})$ and hence

$$\mu_{\Pi_l(X)}(\pi, \hat{1}_l) = \mu_{\Pi_l(\{1,\dots,k\})}(\tilde{\pi}, \hat{1}_l').$$

Therefore we can assume without loss of generality that π has the form $\pi = 1L_1|2L_2|\dots|kL_k$.

Define the labelled set partition $\pi' = 1L_1'|2L_2|\dots|kL_k$, where $L_1' = L_1 \cup \{l\}$ is set, and observe that π' is an element of the interval $I := [\pi, \hat{1}_l]_{\Pi_l(X)}$ with $\pi' > \pi$.

Suppose now that π' has the complement $\sigma \in I$, then $\pi' \vee \sigma = \hat{1}_l$ implies $\sigma = \hat{1}_l$. On the other hand we find $\sigma \wedge \pi' = \pi' > \pi$, so that π' has no complement in the interval I, which contradicts our assumption. Thus we can conclude by Crapo's Theorem 11 that $\mu_{\Pi_l(X)}(\pi, \hat{1}_l) = 0$. □

Definition 19. Let $X = \{1,\dots,n\}$ and define

$$\mu_n = \mu_{\Pi_l(X)}(\hat{0}_l, \hat{1}_l) \tag{8}$$
$$\tilde{\mu}_n = \mu_{\Pi_l(X)}(\sigma_n, \hat{1}_l), \tag{9}$$

where $\sigma_n = 1l|\dots|nl$ denotes the labelled set partition with n labelled singleton blocks.

Example 20. Let $X = \{1,2,3,4,5,6,7\}$ and $\sigma, \pi \in \Pi_l(X)$ with $\sigma = 1l|2l|34|5|67$ and $\pi = 12l|345l|67$. Then we find by Propositions 17 and 9 and Definition 19

$$\begin{aligned}[\sigma,\pi]_{\Pi_l(X)} &\simeq [1l|2l, 12l]_{\Pi_l(\{1,2\})} \times [34|5, 345l]_{\Pi_l(\{3,4,5\})} \times [67, 67]_{\Pi_l(\{6,7\})} \\ &\simeq [1l|2l, 12l]_{\Pi_l(\{1,2\})} \times [3|5, 35l]_{\Pi_l(\{3,5\})} \times [6l, 6l]_{\Pi_l(\{6\})}.\end{aligned}$$

Hence we have $\mu_{\Pi_l(X)}(\sigma,\pi) = \tilde{\mu}_2 \mu_2 \tilde{\mu}_1$.

By Example 20 we conclude, that we only have to consider μ_n and $\tilde{\mu}_n$ to compute the Möbius function $\mu_{\Pi_l(X)}(\sigma,\pi)$ for arbitrary $\sigma, \pi \in \Pi_l(X)$.

Theorem 21. *Let X be a non-empty n-element set. Then*

$$\tilde{\mu}_n = (-1)^{n-1}(n-1)!, \tag{10}$$

$$\mu_n = (-1)^n (n-1)! \tag{11}$$

for all $n \geq 1$.

Proof. Observe that the interval $[\sigma_n, \hat{1}_l]_{\Pi_l(X)}$ is isomorphic to the interval $[\hat{0}, \hat{1}]_{\Pi(X)}$ in the partition lattice $\Pi(X)$. Hence we can conclude by Theorem 13

$$\tilde{\mu}_n = \mu_{\Pi(X)}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!,$$

which proves the first claim.

Now the Möbius function satisfies by Definition 8

$$\sum_{\pi \in \Pi_l(X)} \mu_{\Pi_l(X)}(\pi, \hat{1}_l) = \delta_{\Pi_l(X)}(\hat{0}_l, \hat{1}_l) = 0,$$

with $\delta_{\Pi_l(X)}(\hat{0}_l, \hat{1}_l) = 0$, as $\hat{0}_l \neq \hat{1}_l$, whenever X is a non-empty set. The application of Lemma 18 allows the reduction of the above sum to the non-vanishing Möbius function values and we have

$$\sum_{k=1}^{n} S(n,k)\,(\mu_k + \tilde{\mu}_k) = 0.$$

Here we used the property, that every interval $[\pi, \hat{1}_l]_{\Pi_l(X)}$ with $\pi = B_1L_1|\ldots|B_kL_k$ is isomorphic to the interval $[1L_1|\ldots|kL_k, \hat{1}_l]_{\Pi_l(\{1,\ldots,k\})}$.

Subsequently we show by induction over n that $\mu_n = -\tilde{\mu}_n$. For the basic step $n = 1$ the claim is obviously true. Let $l = n+1$ and assume that the claim is true for all $l \leq n$. Then we have after application of the induction hypothesis to the above sum

$$\begin{aligned}\sum_{k=1}^{n+1} S(n+1,k)\,(\mu_k + \tilde{\mu}_k) &= 0 \\ S(n+1,n+1)\,(\mu_{n+1} + \tilde{\mu}_{n+1}) &= 0 \\ \mu_{n+1} &= -\tilde{\mu}_{n+1},\end{aligned}$$

which proves the claim for $l = n+1$ as well. Hence we have

$$\mu_n = -\tilde{\mu}_n = -1 \cdot (-1)^{n-1}(n-1)!$$

for all $n \geq 1$. □

5 The K-Terminal Reliability

In this section we derive a first splitting approach for the K-terminal reliability $R(G, K)$ at a separating vertex set X in Theorem 38. Furthermore, we examine $R(G, K)$ by defining suitable indicator functions following the ideas presented in the PhD thesis of Tittmann [16].

Definition 22. A *graph* $G = (V, E, \varphi)$ is a triple consisting of a finite set V of *vertices* and a finite set E of *edges* endowed with an incidence function $\varphi\colon E \to V^{(2)}$. Here $V^{(2)}$ denotes the set of the two-element subsets of V.

For convenience of display we often tacitly omit the incidence function φ and just write $G = (V, E)$. Let $F \subseteq E$, then $H = (V, F)$ is a *spanning subgraph* of G and we write $H \subseteq G$.

Definition 23. Let $G = (V, E, \varphi)$ be a graph and $\pi = B_1|\dots|B_k \in \Pi(X)$ with $X \subseteq V$. Then $G_\pi = (V_\pi, E_\pi, \varphi_\pi)$ denotes the π*-merging of* G with vertex set

$$V_\pi = (V \setminus X) \cup \{B_1, \dots, B_k\} \tag{12}$$

and edge set

$$E_\pi = \{e \in E : \varphi(e) \text{ is not a subset of a block in } \pi\}, \tag{13}$$

where the incidence function $\varphi_\pi\colon E_\pi \to V_\pi^{(2)}$ is given by

$$\varphi_\pi(e) = \begin{cases} \{B_i, B_j\} & \text{if } \varphi(e) = \{u, v\}, u \in B_i, v \in B_j, B_i \neq B_j \\ \{B_i, v\} & \text{if } \varphi(e) = \{u, v\}, u \in B_i, v \in V \setminus X \\ \varphi(e) & \text{else.} \end{cases} \tag{14}$$

In other words, G_π denotes the graph that emerges from G by merging all vertices in G that are in a same block in π, where possibly occurring parallel edges are kept and loops are removed. In the case of the one block set partition $\pi = \{X\} = \hat{1}$, we simply write $G_X = (V_X, E_X)$.

Definition 24. Let $G = (V, E)$ be a graph and $p\colon E \to [0, 1], e \mapsto p(e)$. Assume now that the edges $e \in E$ of G are failing independently with the probabilities $q(e) := 1 - p(e)$. We say that the pair (G, p) is a *stochastic network*. In the following we identify the graph G and its corresponding stochastic network (G, p) if there is no danger of confusion. The probability that the spanning subgraph $H = (V, F) \subseteq G$ is realised equals

$$\Pr(H) = \prod_{e \in F} p(e) \prod_{e \in E \setminus F} q(e). \tag{15}$$

Definition 25. A *K-graph* (G, K) is a pair consisting of a graph $G = (V, E)$ and a subset $K \subseteq V$ of *terminal vertices* with $|K| \geq 2$. Every K-graph (G, K) induces a labelled set partition

$$\{(G, K)\} = V_1 L_1 | \dots | V_r L_r \in \Pi_l(V), \tag{16}$$

where two vertices u, $v \in V$ are in a same V_i if and only if u and v are connected in G and we set $L_i = \{l\}$ if $V_i \cap K \neq \emptyset$ and $L_i = \emptyset$ otherwise.

A graph $G = (V, E)$ is *K-connected* if all terminal vertices in K are mutually connected in G or in other words $\{(G, K)\}$ has exactly one labelled block. Finally, we define the *K-connectedness indicator* $M(G, K)$ by

$$M(G, K) = \begin{cases} 1 & G \text{ is } K\text{-connected} \\ 0 & \text{else.} \end{cases} \tag{17}$$

Definition 26. Let (G, K) be a K-graph and (G, p) a stochastic network. We say that (G, K) is a *K-network* and the *K-terminal reliability* $R(G, K)$ is the probability that G is K-connected. Thus

$$R(G, K) = \sum_{H \subseteq G} M(H, K) \Pr(H). \tag{18}$$

Definition 27. Let (G, K) be a K-graph and $G^1 = (V^1, E^1)$, $G^2 = (V^2, E^2)$ subgraphs of G, so that $E^1 \cup E^2 = E$, $E^1 \cap E^2 = \emptyset$, $V^1 \cup V^2 = V$ and $V^1 \cap V^2 = X$ holds. Furthermore we demand that $K^1 := V^1 \cap K \neq \emptyset$ and $K^2 := V^2 \cap K \neq \emptyset$ is satisfied, so that each of the two subgraphs G^1 and G^2 contains at least one terminal vertex.

Under these two conditions (G^1, G^2, X) is said to be a *K-splitting of* (G, K) with *separating vertex set* X.

Definition 28. Define the function $m \colon \Pi_l(X) \to \{0, 1\}, \pi \mapsto m(\pi)$ by

$$m(\pi) = \begin{cases} 1 & \pi \text{ has exactly one labelled block} \\ 0 & \text{else.} \end{cases} \tag{19}$$

Definition 29. Let (G^1, G^2, X) be a K-splitting of (G, K) and define

$$D(G^i, \pi) = \begin{cases} 1 & \{(G^i, K^i)\} \sqcap X = \pi \\ 0 & \text{else} \end{cases} \tag{20}$$

for all $\pi \in \Pi_l(X)$ and $i = 1, 2$.

Definition 30. Let (G^1, G^2, X) be a K-splitting of (G, K). Define for $i = 1, 2$ the K-graphs (G^i_X, K^i_X) with terminal vertex set $K^i_X = (K^i \setminus X) \cup \{X\}$.

Remark 31. Let (G^1, G^2, X) be a K-splitting of (G, K). Observe that the K-connectedness of G implies that every terminal vertex $v \in K^i$ in G^i is connected to at least one vertex in X in G^i for $i = 1, 2$. Note that the above implication can be restated by using the K-connectedness indicator as

$$M(G, K) = 1 \quad \Longrightarrow \quad M(G^1_X, K^1_X)M(G^2_X, K^2_X) = 1. \tag{21}$$

Definition 32. Let (G^1, G^2, X) be a K-splitting of (G, K) and $\pi \in \Pi_l(X)$. The *partition probability* $P(G^i, \pi)$ is defined as

$$P(G^i, \pi) = \sum_{H^i \subseteq G^i} D(H^i, \pi)M(H^i_X, K^i_X)\Pr(H^i) \qquad \text{for } i = 1, 2. \tag{22}$$

Definition 33. Let $\pi = B_1L_1|\dots|B_kL_k \in \Pi_l(X)$ and (G^1, G^2, X) be a K-splitting of (G, K). We say that (G^i_π, K^i_π) for $i = 1, 2$ is the *π-merged K-graph of* (G^i, K^i), where G^i_π denotes the π'-merging of G^i with $\pi' = B_1|\dots|B_k \in \Pi(X)$ and K^i_π the terminal vertex set

$$K^i_\pi = (K^i \setminus X) \cup \{B_i \in \pi' \colon L_i = \{l\} \text{ or } B_i \cap K^i \neq \emptyset\}. \tag{23}$$

Definition 34. Let $X = \{x_1, \dots, x_n\}$ and (G^1, G^2, X) a K-splitting of (G, K). Denote by $\pi_X \in \Pi_l(X)$ the labelled set partition $\pi_X = x_1L_1|\dots|x_nL_n$ with $L_i = \{l\}$ whenever $x_i \in K$ and $L_i = \emptyset$ otherwise.

Furthermore, let $\Pi_l(X, \pi_X)$ be the set of all labelled set partitions $\pi \geq \pi_X$ in $\Pi_l(X)$ with at least one labelled block.

We denote by $P(n, k)$ the number of elements in $\Pi_l(X, \pi_X)$, where we assume that π_X has k labelled and $n - k$ unlabelled blocks.

Theorem 35. *The numbers $P(n, k)$ are given by*

$$P(n, 0) = \sum_{j=1}^{n} \binom{n}{j} B(j)B(n-j),$$
$$P(n, k) = \sum_{j=0}^{n-k} \binom{n-k}{j} B(k+j)B(n-k-j)$$

for $n \geq 1$ and $k \geq 1$.

Proof. Let X be a non-empty n-element set and consider the labelled set partition π_X with k labelled and $n-k$ unlabelled blocks and the number of possible ways to construct a labelled set partition $\sigma \geq \pi_X$ with at least one labelled block.

We can choose in $\binom{n-k}{j}$ different ways j of the $n - k$ unlabelled blocks of π_X being labelled in σ.

Afterwards we have $B(k+j)$ possibilities to partition the labelled blocks and $B(n-k-j)$ choices to partition the remaining unlabelled blocks. In the case $k \geq 1$ there is always at least one labelled block. For $k = 0$ we ensure the existence of at least one labelled block in σ by demanding $j \geq 1$. □

Theorem 36. *Let (G^1, G^2, X) be a K-splitting of (G, K). Then*

$$M(G,K) = \\ M(G^1_X, K^1_X)M(G^2_X, K^2_X) \sum_{\sigma_1,\sigma_2 \in \Pi_l(X,\pi_X)} D(G^1,\sigma_1)m(\sigma_1 \vee \sigma_2)D(G^2,\sigma_2). \quad (24)$$

Proof. Assume first that there exists a terminal vertex $v \in K$, that is not connected to a vertex in X. In this case we have

$$M(G^1_X, K^1_X)M(G^2_X, K^2_X) = 0 \quad (25)$$

and by the contraposition of Remark 31 we find $M(G, K) = 0$ as well, so that the equation is valid in this trivial case.

Hence we can assume now that every terminal vertex $v \in K$ is connected to at least one vertex in X or in other words $M(G^1_X, K^1_X)M(G^2_X, K^2_X) = 1$.

As K^1 and K^2 are non-empty sets, we can ensure that every subgraph G^i induces exactly one labelled set partition ρ_i in X, which has at least one labelled block and we conclude that $\rho_i \in \Pi_l(X, \pi_X)$. Therefore the equation simplifies to

$$M(G,K) = m(\rho_1 \vee \rho_2).$$

Observe now that under the above assumptions G is K-connected if and only if for every two vertices $u, v \in X$, that are connected to a labelled vertex in G, there is a sequence of blocks $B_1, \ldots, B_k \in \rho_1 \cup \rho_2$ with $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \ldots, k-1$, so that $u \in B_1$,$v \in B_k$.

This last characterisation is equivalent to the condition that $\rho_1 \vee \rho_2$ has exactly one labelled block or in other words $m(\rho_1 \vee \rho_2) = 1$. $\square$

Definition 37. Let (G^1, G^2, X) be a K-splitting of (G, K). Denote by $\mathbf{p}(G^i)$ and $\mathbf{r}(G^i)$ the vectors containing the probabilities $P(G^i, \pi)$ and $R(G^i_\pi, K^i_\pi)$ for all $\pi \in \Pi_l(X, \pi_X)$ and define the *transfer matrix* $\mathbf{M}$ as

$$\mathbf{M} = (m(\pi \vee \sigma))_{\substack{\pi \in \Pi_l(X,\pi_X) \\ \sigma \in \Pi_l(X,\pi_X)}}. \quad (26)$$

Theorem 38. *Let (G^1, G^2, X) be a K-splitting of (G, K). Then*

$$R(G,K) = \mathbf{p}(G^1)^T \mathbf{M} \mathbf{p}(G^2). \quad (27)$$

Proof. The K-terminal reliability $R(G, K)$ is by Definition 26

$$R(G,K) = \sum_{H \subseteq G} M(H,K)\mathrm{Pr}(H).$$

The application of Theorem 36 to $M(H, K)$ and the definition of the partition probability yields

$$R(G,K) = \sum_{\sigma_1,\sigma_2 \in \Pi_l(X,\pi_X)} P(G^1,\sigma_1)m(\sigma_1 \vee \sigma_2)P(G^2,\sigma_2),$$

which equals the stated matrix equation. $\square$

In Theorem 39 we give a slight generalisation of a theorem found in the PhD thesis of Tittmann [16].

Theorem 39. *Let* (G^1, G^2, X) *be a* K*-splitting of* (G, K)*. Then*

$$M(G^i_\pi, K^i_\pi) = M(G^i_X, K^i_X) \sum_{\sigma \in \Pi_l(X, \pi_X)} D(G^i, \sigma) m(\pi \vee \sigma) \tag{28}$$

holds for all $\pi \in \Pi_l(X, \pi_X)$ *and* $i = 1, 2$*.*

Proof. First assume that there is a terminal vertex $v \in K^i$, which is not connected to a vertex in X. In this case we have $M(G^i_X, K^i_X) = 0$ and $M(G^i_\pi, K^i_\pi) = 0$ as well, as the graph G^i_π has at least one terminal vertex in X, because $\pi \in \Pi_l(X, \pi_X)$ has at at least one labelled block. Therefore the equation is valid in this trivial case.

Hence we can assume from now on, that every terminal vertex $v \in K^i$ is connected to a vertex in X or in other words $M(G^i_X, K^i_X) = 1$. As $K^i \neq \emptyset$ we conclude that G^i induces in X exactly one labelled set partition ρ with at least one labelled block, that satisfies $\rho \geq \pi_X$. This leaves us with one summand

$$M(G^i_\pi, K^i_\pi) = m(\pi \vee \rho), \tag{29}$$

which is a valid equation by considering the properties of $\pi \vee \rho$ and (G^i_π, K^i_π). □

Theorem 40. *Let* (G^1, G^2, X) *be a* K*-terminal splitting of* (G, K)*. Then*

$$\mathbf{r}(G^i) = \mathbf{M}\mathbf{p}(G^i) \tag{30}$$

holds for $i = 1, 2$*.*

Proof. Let $\pi \in \Pi_l(X, \pi_X)$ and consider the row of the above matrix equation corresponding to π

$$R(G^i_\pi, K^i_\pi) = \sum_{\sigma \in \Pi_l(X, \pi_X)} P(G^i, \sigma) m(\pi \vee \sigma).$$

Observe that this equation holds by Theorem 39, when we consider the definition of the partition probability and a summation over all possible subgraphs. □

6 The Transfer Matrix

In his PhD thesis Tittmann [16] observed that the transfer matrix is generally not invertible. This section states a factorisation of the $\mathbf{M}$ matrix and gives a condition for the existence of $\mathbf{M}^{-1}$ by Corollary 49. This factorisation is then used in the computation of the K-terminal reliability in Section 7. Factorisations of supremum matrices are considered by Wilf [18], Smith [15] and Lindström [11] to solve problems in combinatorics and number theory. Haukkanen and Korkee [10] give further remarks on determinants and inverses of supremum matrices. A nice introduction into the versatile Möbius inversion principle is given by Bender and Goldmann [4].

Theorem 41 (Wilf [18]). *Let P be a finite upper semilattice with $P = \{p_1, \ldots, p_n\}$, so that $p_i \leq p_j$ implies $i \leq j$ and define the upper triangular Zeta-matrix $\mathbf{E} = (e_{ij})$ of format $n \times n$ with entries $e_{ij} = \zeta_P(p_i, p_j)$.*

Furthermore let $f, g\colon P \to \mathbb{R}$ be two functions defining the vectors $\mathbf{f} = (f(p_i))$ and $\mathbf{g} = (g(p_i))$ of length n, so that

$$\mathbf{f} = \mathbf{E}\mathbf{g}$$

is satisfied.

Now denote by $\mathbf{F} = (f_{ij})$ and $\mathbf{G} = (g_{ij})$ matrices of format $n \times n$ with $f_{ij} = f(p_i \vee p_j)$ and $\mathbf{G}$ being a diagonal matrix with entries $g_{ii} = g(p_i)$. Then

$$\mathbf{F} = \mathbf{E}\mathbf{G}\mathbf{E}^T. \tag{31}$$

Remark 42. Assume that the conditions of Theorem 41 are given. We can then compute the vector $\mathbf{g}$ from the vector $\mathbf{f}$ by

$$\mathbf{g} = \mathbf{E}^{-1}\mathbf{f}, \tag{32}$$

which is the *Möbius inversion principle.* Observe that the entries r_{ij} of the matrix $\mathbf{E}^{-1} = (r_{ij})$ satisfy $r_{ij} = \mu_P(p_i, p_j)$.

Definition 43. Assume that the conditions of Theorem 41 are satisfied and define the subset $P_0 \subseteq P$ by

$$P_0 = \{p \in P\colon g(p) \neq 0\}. \tag{33}$$

Furthermore denote by $\mathbf{F}_0$, $\mathbf{E}_0$ and $\mathbf{G}_0$ the matrices emerging form $\mathbf{F}$, $\mathbf{E}$ and $\mathbf{G}$ by the removal of all columns and rows, that are not corresponding to elements in P_0. In general we denote by the bracket notation $[\cdot]_0$ the removal of all rows and columns not corresponding to elements in P_0.

Theorem 44. *Assume that the conditions of Theorem 41 are satisfied. Then the inverse of $\mathbf{F}_0$ exists.*

Proof. By Theorem 41 we have $\mathbf{F} = \mathbf{E}\mathbf{G}\mathbf{E}^T$ and therefore

$$f_{ij} = \sum_{k=1}^{n} e_{ik} g_{kk} e_{jk} = \sum_{k : p_k \in P_0} e_{ik} g_{kk} e_{jk},$$

for all $p_i, p_j \in P_0$, which gives $\mathbf{F}_0 = \mathbf{E}_0\mathbf{G}_0\mathbf{E}_0^T$. Observe now that the inverses of $\mathbf{E}_0$ and $\mathbf{G}_0$ exist, as $\mathbf{E}_0$ represents the Zeta-function of the subposet P_0 and $\mathbf{G}_0$ is a diagonal matrix with non-vanishing diagonal elements. $\square$

Remark 45. First observe that the set $\Pi_l(X, \pi_X)$ is a finite upper semilattice. In the following we apply Theorem 44 to the transfer matrix $\mathbf{M}$ and the function g in Theorem 41 is denoted by λ. Hence we have by Remark 42

$$\lambda(\pi) = \sum_{\substack{\sigma \in \Pi_l(X, \pi_X) \\ \sigma \geq \pi}} \mu_{\Pi_l(X)}(\pi, \sigma) m(\sigma) \tag{34}$$

for all $\pi \in \Pi_l(X, \pi_X)$.

Definition 46. Let (G^1, G^2, X) be a K-splitting of (G, K). Denote by $\mathbf{\Lambda}$ the diagonal matrix with entries $\lambda_{\pi,\pi} = \lambda(\pi)$ for all $\pi \in \Pi_l(X, \pi_X)$ and by $\mathbf{Z}$ the matrix with entries $z_{\pi,\sigma} = [\pi \leq \sigma]$ for all $\pi, \sigma \in \Pi_l(X, \pi_X)$.

Corollary 47. *We have*

$$\mathbf{M} = \mathbf{Z}\mathbf{\Lambda}\mathbf{Z}^T \quad \text{and} \quad \mathbf{M}_0 = \mathbf{Z}_0\mathbf{\Lambda}_0\mathbf{Z}_0^T,$$

where $\mathbf{M}_0^{-1}$ *always exists.*

Definition 48. Let X be an n-element set and assume that π_X has k labelled and $n-k$ unlabelled blocks. Denote now by $\Pi_l(X, \pi_X)_0$ the set

$$\Pi_l(X, \pi_X)_0 = \{\pi \in \Pi_l(X, \pi_X) : \lambda(\pi) \neq 0\} \tag{35}$$

and the number of elements in $\Pi_l(X, \pi_X)_0$ by $P_0(n, k)$.

Corollary 49. *The matrix* $\mathbf{M}$ *is invertible if and only if* $P(n, k) = P_0(n, k)$ *is satisfied. In this case we have* $\mathbf{M} = \mathbf{M}_0$.

Definition 50. Let $\pi \in \Pi_l(X, \pi_X)$ and denote by $\pi^* \in \Pi_l(X, \pi_X)$ the smallest labelled set partition with at most one labelled block, so that $\pi^* \geq \pi$. In other words π^* emerges from π by the union of all labelled blocks in π.

Theorem 51. *The set* $\Pi(X, \pi_X)_0$ *consists of all labelled set partitions* π *in* $\Pi_l(X, \pi_X)$ *with at most one unlabelled block. We have even more*

$$\lambda(\pi) = \mu_{\Pi_l(X)}(\pi, \pi^*) \tag{36}$$

for all $\pi \in \Pi(X, \pi_X)_0$.

Proof. Partition π by setting $\pi = \rho \cup \tau$, where $\rho \in \Pi_l(L)$ and $\tau \in \Pi(U)$ are consisting of all labelled and all unlabelled blocks of π, respectively. Hence we have $L \cup U = X$ and $L \cap U = \emptyset$. It is by definition

$$\lambda(\pi) = \sum_{\sigma \in \Pi_l(X)} \mu(\pi, \sigma) m(\sigma).$$

Now the condition $\mu_{\Pi_l(X)}(\pi, \sigma) \neq 0$ implies by Lemma 18 that $\sigma \geq \pi$ has the form $\sigma = \varepsilon \cup \omega$ where $\varepsilon \in \Pi_l(L)$ and $\tau \in \Pi(U)$.

Therefore we can write

$$\begin{aligned}
\lambda(\pi) &= \sum_{\substack{\varepsilon \in \Pi_l(L) \\ \omega \in \Pi(U)}} \mu_{\Pi_l(X)}(\rho \cup \tau, \varepsilon \cup \omega) m(\varepsilon \cup \omega) \\
&= \sum_{\varepsilon \in \Pi_l(L)} \sum_{\omega \in \Pi(U)} \mu_{\Pi_l(L)}(\rho, \varepsilon) \mu_{\Pi(U)}(\tau, \omega) m(\varepsilon \cup \omega),
\end{aligned}$$

where the last line follows by the application of Proposition 9. Note that $m(\varepsilon \cup \omega) = 1$ if and only if $\varepsilon = \rho^*$ for all $\varepsilon \in \Pi_l(L)$ and $\omega \in \Pi(U)$.

Hence

$$\begin{aligned}
\lambda(\pi) &= \mu_{\Pi_l(L)}(\rho, \rho^*) \sum_{\omega \in \Pi(U)} \mu_{\Pi(U)}(\tau, \omega) \\
&= \mu_{\Pi_l(L)}(\rho, \rho^*) \delta_{\Pi(U)}(\tau, \hat{1}),
\end{aligned}$$

where the last line follows from the definition of the Möbius function. This proves the claim, as $\mu_{\Pi_l(L)}(\rho, \rho^*)$ is due to Theorem 21 non-vanishing. $\square$

Theorem 52. *The numbers $P_0(n, k)$ are given by*

$$P_0(n, 0) = \sum_{j=0}^{n-1} \binom{n}{j} B(n-j) = B(n+1) - 1, \tag{37}$$

$$P_0(n, k) = \sum_{j=0}^{n-k} \binom{n-k}{j} B(n-j) \tag{38}$$

for $n, k \geq 1$.

Proof. Assume that π_X is a labelled set partition with k labelled and $n-k$ unlabelled blocks. We count all labelled set partitions $\sigma \geq \pi_X$ with at most one unlabelled block and at least one labelled block.

Now we can choose in $\binom{n-k}{j}$ ways a possibly empty set of the unlabelled blocks in π_X, that contribute to the possibly non-existing ($j = 0$) unlabelled block in the labelled set partition σ. Furthermore we can partition the remaining $n-j$ elements in $B(n-j)$ different ways, which gives the contribution of the labelled blocks of σ. Observe that we have to ensure in the case $k = 0$, that there is at least one labelled block after all, which gives the condition $j \leq n-1$ in the sum of $P_0(n, 0)$. $\square$

7 The Splitting Formula

This section states Theorem 55, which is the centrepiece of this article.

Lemma 53. *Let (G^1, G^2, X) be a K-splitting of (G, K). Then*

$$R(G,K) = \left[\mathbf{Z}^T \mathbf{p}(G^1)\right]_0^T \mathbf{\Lambda}_0 \left[\mathbf{Z}^T \mathbf{p}(G^2)\right]_0 . \tag{39}$$

Proof. By Theorem 38 we have

$$R(G,K) = \mathbf{p}(G^1)^T \mathbf{M} \mathbf{p}(G^2)$$

and the factorisation of the $\mathbf{M}$ matrix yields

$$R(G,K) = \mathbf{p}(G^1)^T \mathbf{Z} \mathbf{\Lambda} \mathbf{Z}^T \mathbf{p}(G^2).$$

Considering only the non-vanishing elements of the diagonal matrix $\mathbf{\Lambda}$ gives then

$$R(G,K) = \left[\mathbf{Z}^T \mathbf{p}(G^1)\right]_0^T \mathbf{\Lambda}_0 \left[\mathbf{Z}^T \mathbf{p}(G^2)\right]_0 . \qquad \square$$

Lemma 54. *Let (G^1, G^2, X) be a K-splitting of (G, K). Then*

$$\left[\mathbf{Z}^T \mathbf{p}(G^i)\right]_0 = \mathbf{\Lambda}_0^{-1} \mathbf{Z}_0^{-1} \mathbf{r}_0(G^i) \quad \textit{for } i = 1, 2. \tag{40}$$

Proof. By Theorem 40 we have the equation

$$\mathbf{M} \mathbf{p}(G^i) = \mathbf{r}(G^i)$$

and the factorisation of the $\mathbf{M}$ matrix yields

$$\mathbf{\Lambda} \mathbf{Z}^T \mathbf{p}(G^i) = \mathbf{Z}^{-1} \mathbf{r}(G^i).$$

Considering only the non-vanishing elements of the diagonal matrix $\mathbf{\Lambda}$ gives

$$\mathbf{\Lambda}_0 \left[\mathbf{Z}^T \mathbf{p}(G^i)\right]_0 = \left[\mathbf{Z}^{-1} \mathbf{r}(G^i)\right]_0 .$$

Note that we have the equality

$$\left[\mathbf{Z}^{-1} \mathbf{r}(G^i)\right]_0 = \mathbf{Z}_0^{-1} \mathbf{r}_0(G^i).$$

Hence we can conclude

$$\mathbf{\Lambda}_0 \left[\mathbf{Z}^T \mathbf{p}(G^i)\right]_0 = \mathbf{Z}_0^{-1} \mathbf{r}_0(G^i),$$

which leads after multiplication with $\mathbf{\Lambda}_0^{-1}$ to the desired result. $\square$

Theorem 55. *Let (G^1, G^2, X) be a K-splitting of (G, K). Then*

$$R(G,K) = \mathbf{r}_0(G^1)^T \mathbf{M}_0^{-1} \mathbf{r}_0(G^2). \tag{41}$$

Proof. By Lemma 53 we have

$$R(G,K) = \left[\mathbf{Z}^T \mathbf{p}(G^1)\right]_0^T \mathbf{\Lambda}_0 \left[\mathbf{Z}^T \mathbf{p}(G^2)\right]_0 ,$$

whereas Lemma 54 gives

$$\left[\mathbf{Z}^T \mathbf{p}(G^i)\right]_0 = \mathbf{\Lambda}_0^{-1} \mathbf{Z}_0^{-1} \mathbf{r}_0(G^i)$$

for $i = 1, 2$. Substituting the second equation in the first one proves the claim

$$\begin{aligned} R(G,K) &= \mathbf{r}_0(G^1)^T \mathbf{Z}_0^{-1T} \mathbf{\Lambda}_0^{-1T} \mathbf{\Lambda}_0 \mathbf{\Lambda}_0^{-1} \mathbf{Z}_0^{-1} \mathbf{r}_0(G^2) \\ &= \mathbf{r}_0(G^1)^T [\mathbf{Z}_0^T \mathbf{\Lambda}_0 \mathbf{Z}_0]^{-1} \mathbf{r}_0(G^2) \\ &= \mathbf{r}_0(G^1)^T \mathbf{M}_0^{-1} \mathbf{r}_0(G^2). \end{aligned} \qquad \square$$

8 Conclusion

The splitting formula stated by Theorem 55 shows that the K-terminal reliability can be computed for graphs with small separating vertex sets. We showed that our approach is superior to former known methods by achieving a tremendous reduction of the necessary states by utilising the factoring of the transfer matrix.

Even more we proved that the computational efficiency can be further improved by using separating vertex sets containing terminal vertices.

Observe that we could easily extend our approach to a recursive decomposition scheme by the transfer-matrix method. This extension leads to a polynomial time algorithm for the computation of $R(G, K)$ for graphs with restricted treewidth.

Finally, the splitting approach has an amendable form, as it expresses the K-terminal reliability as a sum of linear combinations of K-terminal reliabilities of the subgraphs of the splitting.

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